Weakly-Nonlinear, Long Internal Gravity Waves in Stratified Fluids of Finite Depth

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This paper presents an analytical investigation of the propagation of a weakly-nonlinear, long internal gravity wave in a stratified medium of finite total depth. The governing equation is derived and shown to reduce to the KdV equation in the shallow-water limit and to the Benjamin/Ono equation in the deep-water limit. The equation is also shown to possess four conserved quantities, just as was the case for the deep-water waves considered by Ono.

The equation suggests the existence of a steady-state waveform described by two parameters which degenerates into the one-parameter, steady-state waveforms discussed by Benjamin. A numerical approach using Fornberg's pseudospectral method is used to examine the solution. The results demonstrate the existence of a solitary wavelike steady-state solution with a solitonlike behavior. The effect of finite water depth on the waveform and the wave speed of the steady-state solitary waves is discussed.

Nomenclature

a	= maximum amplitude of internal wave displacements in vertical direction
a_1, b_0, b_1	= constants appearing in outer expansion for $\phi(z)$
$A(x,t)$	= amplitude of internal wave stream function
A_m	= maximum or reference value of $A(x,t)$
$c(k)$	= phase speed
c_0	= $c(0)$
$F(\hat{x},\hat{H}_1)$	= functional dependence of normalized stream function amplitude on normalized x coordinate and normalized depth
g	= acceleration due to gravity
$G(x)$	= kernel function appearing in nonlinear wave equation
h	= characteristic half-thickness of pycnocline
H, H_1, H_2	= total fluid depth and depths of fluid above and below pycnocline
$H(x)$	= Heaviside unit step function
k	= wave number
$K(x)$	= kernel function appearing in nonlinear wave equation
ℓ	= characteristic length used in normalization of the internal wave equation
t	= time
$T\{\ \}$	= integral transformation defined in Sec. III
u, w	= horizontal and vertical fluid velocities caused by passage of internal wave
x, z	= two-dimensional coordinate system, with x measured positive horizontally in direction of wave propagation and z measured positive vertically upward from pycnocline
α	= coefficient of nonlinear term
β	= coefficient of dispersive term in the infinite- and finite-depth internal wave equations
γ	= coefficient of dispersive term in the shallow-water internal wave equation
$\Gamma(\hat{H}_1)$	= functional dependence of normalized solitary wave speed on normalized depth
δ	= ratio of pycnocline thickness to total fluid depth (h/H)
$\delta(x)$	= Dirac delta function
ϵ	= a/h amplitude parameter
λ	= characteristic wave length
$\Lambda(\hat{H}_1)$	= functional dependence of normalized wavelength of a solitary wave on normalized depth
μ	= wavelength parameter = h/λ
ξ	= $x - c_0 t$ (or integration variable comparable to x; see Eq. (3a) as an example)
ρ	= fluid density
$\rho_0(z)$	= undisturbed vertical density profile
ρ_1, ρ_2	= fluid densities above and below pycnocline
σ	= $\Delta\rho/\rho$, the Boussinesq parameter
τ	= slow-time variable
ϕ	= vertical mode eigenfunction
ψ	= stream function; $\psi(x,z,t) = A(x,t)\,\phi(z)$
ω	= radian frequence of internal wave
$(\tilde{\ })$	= quantity representative of the inner region asymptotic expansion; dimensionless quantity, scaled with respect to the parameter h
$(\hat{\ })$	= normalized quantity, scaled with respect to both amplitude and wavelength parameters so that similarity solutions may be obtained (see Sec. IVA)

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I. Introduction

THE existence of a class of nonlinear waves of permanent form in a density-stratified medium has been demonstrated by several investigators (notably Keulegan,[6] Long,[7,8] Benjamin[1,2] Benney,[3] Davis and Acrivos,[4] and Ono[9]). The mechanism whereby these waves, usually known as solitary waves, retain their shape and speed can be explained in terms of a balance between nonlinear and dispersive effects. For very long waves, dispersive effects are weak, and the wave propagation is similar to that in a nonlinear-nondispersive medium, such as waves in a compressible fluid. In such a medium, the propagation speed of a disturbance is dependent on the magnitude of the disturbance, so, typically, one side of the waveform steepens as the wave propagates. The wave will eventually break, unless this steepening is counteracted by effects which tend to smooth out the large gradients. On the other hand, in the case of very small disturbances, the dispersive effects dominate and the propagation speed essentially depends upon the wavelength.

These waves tend to spread out as they propagate. Therefore, the solitary wave can be thought of as a special case in which the nonlinear steepening effects are exactly balanced by the dispersive effects. The existence of a balance implies a definite relationship between the speed, amplitude, and wavelength of the waveform. Moreover, since dispersive effects vary with the depth of the medium, the existing analyses can be divided into two basic groups: shallow-water theory and deep-water theory.

For the case where the wavelength is long compared to the total water depth, i.e., shallow-water theory, the steady-state version of the theory was presented by Benjamin.[1] The equation which describes the unsteady motion of weakly nonlinear gravity waves in stratified fluid was derived by Benney.[3] Because of the long-wave assumption used in these analyses, the characteristic length scale of the pycnocline (the region where the density varies) may be assumed to be the same as the total depth H. In terms of the undisturbed density distribution, $\rho_0(z)$, and the stream function perturbation $\psi = A(x,t)\phi(z)$, Benney's analysis yields

$$A_t + c_0 A_x + \alpha A A_x + \gamma A_{xxx} = 0 \qquad \text{(1a)}$$

where

$$\alpha = \frac{3}{2} \frac{\int_0^H \rho_0 \phi'^3 \mathrm{d}z}{\int_0^H \rho_0 \phi'^2 \mathrm{d}z} \qquad \text{(1b)}$$

and

$$\gamma = \frac{c_0}{2} \frac{\int_0^H \rho_0 \phi^2 \mathrm{d}z}{\int_0^H \rho_0 \phi'^2 \mathrm{d}z} \qquad \text{(1c)}$$

The eigenfunction $\phi(z)$ is determined from

$$(\rho_0 \phi')' - \frac{g}{c_0^2} \rho_0' \phi = 0 \qquad \text{(2a)}$$

with boundary conditions

$$\phi(0) = \phi(H) = 0 \qquad \text{(2b)}$$

Here, the prime superscript (′) denotes differentiation with respect to z.

By defining an amplitude parameter, $\epsilon = a/h$, and a wavelength parameter, $\mu = h/\lambda$, it can be shown that the ratio of the nonlinear term to the dispersive term is $0(\epsilon/\mu^2)$. Therefore, nonlinear and dispersive effects are comparable, and solitary waves are possible, when $a\lambda^2/h^3 = 0(1)$.

The corresponding equation for the infinite depth case derived by Ono[9] is:

$$A_t + c_0 A_x + \alpha A A_x - \beta \frac{\partial^2}{\partial x^2} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{A(\xi,t)\mathrm{d}\xi}{x-\xi} = 0 \qquad \text{(3a)}$$

where

$$\beta = \frac{\frac{1}{2} c_0 \rho_\infty}{\int_0^h \rho_0 \phi'^2 \mathrm{d}z} \qquad \text{(3b)}$$

The integral in Eq. (3a) is taken for the principal value only and ρ_∞ denotes the fluid density far from the pycnocline. The quantity α is given by Eq. (1b), and the eigenfunction ϕ satisfies the same homogeneous equation, Eq. (2a), but with somewhat different boundary conditions:

$$\phi(0) = 0, \qquad \phi'(h) = 0 \qquad \text{(3c)}$$

Here, h is much less than the total water depth. It should be noted that the formulation cited here is for the case where the density variation is concentrated in a thin layer of thickness h adjacent to a boundary $z = 0$ in an infinitely deep medium. A slight modification is required for other models of density distribution. It should also be noted that the nonlinear and dispersive effects are comparable when $\epsilon/\mu = a\lambda h^2 \sim 0(1)$, as compared to $a\lambda^2/h^3 \sim 0(1)$ for the shallow-water case. Since $\lambda/h \gg 1$, the difference in the two orderings implies that for permanent waveforms of the same wavelength, either a much larger wave amplitude or a much smaller pycnocline thickness is required for the deep-water case than for the shallow-water case. In other words, qualitatively, the waves need to be "more nonlinear" for the same ratio of λ/h in order to achieve a balance. This "enhanced" wave dispersiveness for the deep-water case can be physically understood when one realizes that the wave dispersion is a result of including the vertical acceleration of fluid particles. In shallow water, only a thin layer of fluid moves vertically, whereas, in deep water, a large body of fluid of the order of the wavelength is disturbed. Therefore, dispersion is expected to be stronger in the deep-water case.

Both of these theories have been studied extensively, and certain differences and similarities of the respective solutions have been noted. The form of the first three terms in Eqs. (1a) and (3a) is the same, so the essential difference between the two cases can be traced to the behavior of the dispersive term, or equivalently, to the behavior of the dispersion relation. In particular, we are interested in the dependence of the frequency, ω, as a function of the wave number, k, for $k \ll 1$ (long waves). For the shallow-water case, the dispersion relation is:

$$\omega = k(c_0 - \gamma k^2 + \ldots) \qquad \text{(4)}$$

while for the deep-water case the dispersion relation is

$$\omega = k(c_0 - \beta |k| + \ldots) \qquad \text{(5)}$$

Note that the water depth, which can be expressed in terms of the wavelength, never appears explicitly because Eqs. (4) and (5) only apply to the cases $H/\lambda \ll 1$ and $H\lambda \gg 1$, respectively. Thus, neither of these theories can predict the wave propagation at depths where $H/\lambda = 0(1)$.

In this paper, we present a more general theory which is applicable to water of arbitrary depth. In Sec. II we formulate the governing equation for the case of a long, weakly nonlinear wave propagating along a thin pycnocline located within water of finite total depth, and show that this equation may be extended to describe the case wherein the pycnocline is adjacent to either the upper or lower boundary. In Sec. III, we show that the governing equation possesses four conserved quantities which provide some integral constraints on the solutions. Finally, in Sec. IV, we present and discuss some numerical results for the case in which the pycnocline is located exactly at mid-depth.

II. Formulation

A. Problem Description

We are interested in the motion of a nonlinear internal gravity wave within a horizontally stratified medium having a representative density distribution as shown in Fig. 1. In the undisturbed fluid, the density variation is confined to a thin layer of thickness h, located at a distance H_1 below the upper surface, with the total depth defined as $H = H_1 + H_2$. The

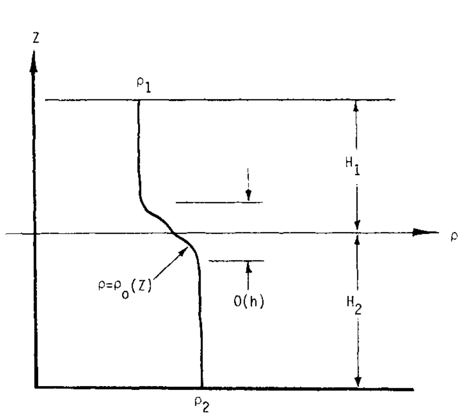

Fig. 1 Representative density distribution.

origin of the coordinate axis is chosen to be at the center of the undisturbed pycnocline, with the positive z axis pointing upward. We will restrict our investigation to weakly nonlinear internal waves with wavelength λ much larger than the characteristic scale for the density profile h. Initially, we will also assume H/λ to be of order unity, but this condition will be modified in later subsections to show how the cases of deep water ($H/\lambda \gg 1$) and shallow water ($h/\lambda \ll 1$) are recovered.

The derivation presented in this report follows the formulation given by Whitham.[10] If the linear dispersion relation is given by

$$\omega = kc(k)$$

with $c(0) = c_0$ (a finite constant), then the equation for weakly-nonlinear waves is given by

$$A_t + AA_x + \frac{\partial}{\partial x}\int_{-\infty}^{\infty} K(x-\xi)A(\xi,t)\mathrm{d}\xi = 0 \tag{6a}$$

where

$$K(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} c(k)e^{ikx}\mathrm{d}k \tag{6b}$$

Equation (6b) represents the nonlinear effect, while the third term results from the dispersive nature of the medium. In Sec. IIB, the effect of nonlinearity is first derived for nondispersive waves. The derivation of the linear dispersion relation is discussed in Sec. IIC. The results of these two subsections are then combined in Sec. IID to construct an equation in the form of Eq. (6). Reductions of the resulting equation to the special extreme cases of deep-water and shallow-water internal waves are demonstrated in the subsequent subsections.

B. Effect of Nonlinearity for Nondispersive Waves

The effect of a weak nonlinearity on wave propagation in a pycnocline was derived for shallow water by Benney[3] and for deep water by Ono[9]; both authors treated the effects of nonlinearity and dispersion simultaneously. Here, we will use some ideas from each of these analyses, but we will focus our attention only on the nonlinearity. Furthermore, we will remove any restrictions on the depth, except for the assumption that it is large compared to the pycnocline thickness (i.e., $\delta = h/H \ll 1$). To isolate the nonlinear effect, we first introduce the scaled variables

$$x^* = x/\lambda,\ z^* = z/h,\ t^* = t\sqrt{gh}/\lambda$$

$$u^* = u/\sqrt{gh},\ w^* = w/(\mu\sqrt{gh})$$

$$\rho^* = \rho/\rho_0, p^* = p/(\rho_0 gh)$$

where ρ_0 is a reference density and $\mu = h/\lambda$. After substitution in the equations of two-dimensional, incompressible stratified flow and taking the limit $\mu \to 0$, the remaining terms are ordered only by $\epsilon = a/h$, $\delta = h/H$, and $\sigma = \Delta\rho/\rho_0$, and the equations of motion become:

$$\rho_t + \psi_z\rho_x - \psi_x\rho_z = 0$$

$$\rho(\psi_{zt} + \psi_z\psi_{zx} - \psi_x\psi_{xz}) + p_x = 0$$

$$p_z + \rho g = 0 \tag{7}$$

where ψ is the streamfunction. Since the effects of a weak nonlinearity appear only on a long time scale, we introduce a slow-time variable $\tau = \epsilon t$ and assume the following expansions:

$$\psi = \epsilon\psi_1 + \epsilon^2\psi_2 + \ldots$$

$$\rho = \rho_0 + \epsilon\rho_1 + \epsilon^3\rho_2 + \ldots$$

$$p = p_0 + \epsilon p_1 + \epsilon^2 p_2 + \ldots \tag{8}$$

For waves which propagate in one direction into an initially undisturbed region, the first-order streamfunction is:

$$\psi_1 = A(\xi,t)\phi(z),\ \xi = x - c_0 t$$

and the $0(\epsilon)$ equations yield

$$u_1 = A(\xi,\tau)\phi'(z)$$

$$w_1 = -A_\xi(\xi,\tau)\phi(z)$$

$$\rho_1 = -\frac{\rho_0'(z)}{c_0}A(\xi,\tau)\phi(z)$$

$$p_1 = c_0\rho_0(z)A(\xi,\tau)\phi(z) \tag{9}$$

Note that the time variation of the amplitude A can be neglected in the $\xi - \tau$ coordinate to this order. The equation for the eigenfunction ϕ is:

$$(\rho_0\phi')' - \frac{g}{c_0^2}\rho_0\phi = 0 \tag{10a}$$

The boundary conditions for ϕ are, to this order,

$$\phi(H_1) = \phi(-H_2) = 0 \tag{10b}$$

Equation (10a), together with the homogenous boundary conditions (Eq. 10b), constitutes a standard eigenvalue problem which may be solved by a variety of methods. A detailed discussion of the eigenvalue, c_0, and the eigenfunction, $\phi(z)$, is given in Sec. IIC. For the present purpose of considering the nonlinear effect, it is adequate to assume that both c_0 and ϕ are known.

Proceeding to $0(\epsilon^2)$ terms where the effects of nonlinearity first appear, and using the properties of the eigenfunction, ϕ, we obtain the equation for the amplitude $A(\xi,\tau)$ as

$$A_\tau + \alpha AA_\xi = 0 \tag{11a}$$

where

$$\alpha = \frac{3}{2}\frac{\int_{-H_1}^{H_2}\rho_0\phi'^3\mathrm{d}z}{\int_{-H_2}^{H_1}\rho_0\phi'^2\mathrm{d}z} \tag{11b}$$

which is the same as the form given by Benney. These equations provide the form of the nonlinear term for all subsequent derivations wherein the dispersive effects are considered.

C. Effects of Dispersive Terms for Linear Waves

In this portion of the analysis, we isolate the dispersive terms by expanding the variables for small ϵ, as in Eq. (8), and taking the limit $\epsilon \to 0$. We find that we must retain some of the vertical acceleration terms that were neglected in obtaining Eqs. (7) and that the vertical mode equation becomes

$$(\rho_0 \phi')' - [(g/c^2)\rho_0' + k^2 \rho_0]\phi = 0 \tag{12}$$

with the boundary conditions given by Eq. (10b) and where now $c = c(k)$. A matched asymptotic expansion procedure will be used to find the functional form of $c(k)$ and the associated eigenfunction.

In the outer region, away from the pycnocline, $\rho_0' \sim 0$, so Eq. (12) becomes

$$\phi'' - k^2 \phi = 0 \tag{13}$$

with $\phi(H_1) = \phi(-H_2) = 0$.

Within the pycnocline (the inner region), we introduce the inner variables $\tilde{z} = z/h$, $\tilde{\rho}_0(\tilde{z}) = \rho_0(z)$, and $\tilde{\phi}(\tilde{z}) = \phi(z)$. Equation (12) becomes

$$(\tilde{\rho}_0 \tilde{\phi}')' - \frac{gh}{c^2}\tilde{\rho}_0' \tilde{\phi} = k^2 h^2 \tilde{\rho}_0 \tilde{\phi} \tag{14}$$

with boundary conditions to be determined through matching with the outer solution. For this purpose, we assume that $\delta = h/H$ is small and that the inner solution can be expanded as

$$\tilde{\phi}(\tilde{z}) \sim \tilde{\phi}_0(\tilde{z}) + \delta\tilde{\phi}_1(\tilde{z}) + \ldots, -\infty < \tilde{z} < \infty$$

$$\frac{gh}{c^2} \sim \nu_0 + \delta\nu_1 + \ldots$$

The corresponding differential equations are

$$O(1): \ (\tilde{\rho}_0 \tilde{\phi}_0')' - \nu_0 \tilde{\rho}_0 \tilde{\phi}_0 = 0 \tag{15}$$

$$O(\delta): \ (\rho_0 \phi_1')' - \nu_0 \tilde{\rho}_0 \tilde{\phi}_1 = \nu_1 \tilde{\rho}_0' \tilde{\phi}_0 \tag{16}$$

The corresponding boundary conditions must be contained from the outer solution of Eq. (13), which can be written as

$$\phi(z) \sim (1 + \delta a_1 + \ldots)\frac{\sinh k(H_1 - z)}{\sinh kH_1}, 0 < z \leq H_1 \tag{17a}$$

$$\phi(z) \sim (b_0 + \delta b_1 + \ldots)\frac{\sinh k(H_2 + z)}{\sinh kH_2}, -H_2 < z < 0 \tag{17b}$$

Therefore,

$$\left.\begin{aligned}\tilde{\phi}_0 &= 1\\ \tilde{\phi}_1 &\sim a_1 - zkH\coth kH_1\end{aligned}\right\}\tilde{z} \to +\infty$$

$$\left.\begin{aligned}\tilde{\phi}_0 &= b_0\\ \tilde{\phi}_1 &\sim b_1 + b_0 \tilde{z} kH \coth kH_2\end{aligned}\right\}\tilde{z} \to -\infty \tag{18}$$

Here, ϕ has been normalized so that $\tilde{\phi} \to 1$ as $\tilde{z} \to +\infty$. Note that the right-hand side of Eq. (14) has no effect on the solution to this order.

By multiplying Eq. (15) by $\tilde{\phi}_0$ and integrating, we obtain

$$\nu_0 = -\frac{\int_{-\infty}^{\infty} \tilde{\rho}_0 \tilde{\phi}_0'^2 \mathrm{d}z}{\int_{-\infty}^{\infty} \rho_0' \phi_0^2 \mathrm{d}\tilde{z}} \tag{19}$$

Similarly, from Eq. (16) we obtain

$$\nu_1 = -\frac{\rho_1 kH \coth kH_1 + \rho_2 b_0^2 kH \coth kH_2}{\int_{-\infty}^{\infty} \tilde{\rho}_0' \tilde{\phi}_0^2 \mathrm{d}\tilde{z}} \tag{20}$$

where

$$\rho_1 = \tilde{\rho}_0(+\infty) = \rho_0(H_1), \rho_2 = \tilde{\rho}_0(-\infty) = \rho_0(-H_2)$$

Hence

$$c(k) \sim \sqrt{\frac{gh}{\nu_0}}\left[1 - \frac{\delta\nu_1}{2\nu_0}\right] \sim c_0\left[1 - \beta_1\left(k\coth kH_1 - \frac{1}{H_1}\right) - \beta_2\left(k\coth kH_2 - \frac{1}{H_2}\right)\right] \tag{21}$$

where

$$c_0 = \sqrt{\frac{gh}{\nu_0}}\left[1 - \frac{\beta_1}{H_1} - \frac{\beta_2}{H_2}\right] \tag{22}$$

and

$$\beta_1 = \frac{\frac{1}{2}\rho_1 h}{\int_{-\infty}^{\infty} \tilde{\rho}_0 \tilde{\phi}_0'^2 \mathrm{d}\tilde{z}} \qquad \beta_2 = \frac{\rho_2}{\rho_1}\beta_1 b_0^2 \tag{23}$$

As defined in Eq. (6), the kernel $K(x)$ corresponding to Eq. 21 is:

$$k(x) = c_0 \frac{\mathrm{d}}{\mathrm{d}x}\left\{H(x) - \frac{\beta_1}{2H_1}\left(\coth\frac{\pi x}{2H_1} - \mathrm{sgn} x\right) - \frac{\beta_2}{2H_2}\left(\coth\frac{\pi x}{2H_2} - \mathrm{sgn} x\right)\right\} \tag{24}$$

where $H(x)$ is the Heaviside unit step function.

D. The Governing Equations for Weakly Nonlinear Long Internal Waves in a Finite Depth Fluid

Combining the results of the previous two subsections, we obtain the following nonlinear integro-differential equation:

$$A_t + c_0 A_x + \alpha A A_x - c_0 \frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty} G(x - \xi) A(\xi, \tau)\mathrm{d}\xi = 0 \tag{25a}$$

where

$$G(x) = \frac{\beta_1}{2H_1}\left[\coth\frac{\pi x}{2H_1} - \mathrm{sgn}(x)\right] + \frac{\beta_2}{2H_2}\left[\coth\frac{\pi x}{2H_2} - \mathrm{sgn}(x)\right] \tag{25b}$$

and $c(k)$ is given by Eq. (21), c_0 by Eq. (22), β_1 and β_2 by Eq. (23), and α by Eq. (11b). It should be noted that the same relations are obtained if a rigorous expansion in the four parameters ϵ, μ, δ, and σ is used.

E. Special Cases

1. Thin Pycnocline Located Near a Boundary

In the limit $H_2 \to 0$, it is easily shown that the equation for A becomes

$$A_t + c_0 A_x + \alpha A A_x = \frac{\beta_1 c_0}{2H_1} \frac{\partial^2}{\partial x^2} \int_{-\infty}^{\infty} \left[\coth \frac{\pi(x-\xi)}{2H_1} - \text{sgn}(x-\xi)\right] A(\xi,\tau) d\xi \quad (26)$$

If $H_1 \to \infty$, the right-hand side of Eq. (26) becomes

$$\frac{\beta_1 c_0}{\pi} \frac{\partial^2}{\partial x^2} \int_{-\infty}^{\infty} \frac{A(\xi,\tau)}{x-\xi} d\xi$$

and, in the expressions for the parameters, the domain of integration is changed to $[0,\infty]$. This is the same form as derived by Ono.

It is interesting to note that the phase speed, Eq. (21) demonstrates the same behavior in both the short-wave limit (large k) and the infinite-depth limit. In either case, we have

$$c(k) = \text{constant} - c_0 \beta_1 |k|$$

This indicates that, in water of finite depth, waves which are short compared to the total depth behave similarly to long waves in infinitely deep water.

2. Shallow-Water Case (KdV/Benney)

As kH_1, $kH_2 \to 0$, the depth becomes small compared to the wavelength and $c(k)$, as given by Eq. (21) becomes

$$c(k) \sim c_0 \left(1 - \frac{1}{3} \beta_1 H k^2 + \ldots\right) \quad (27)$$

and Eq. (26) becomes

$$A_t + c_0 A_x + \alpha A A_x + \frac{1}{3} \beta_1 H c_0 A_{xxx} = 0 \quad (28)$$

We will now show that this agrees with the previous long-wave analyses. As derived by Benney, the equation for the shallow-water case is:

$$A_t + c_0 A_x + \alpha A A_x + \gamma A_{xxx} = 0 \quad (29a)$$

where

$$\gamma = \frac{c_0}{2} \frac{\int_0^H \rho_0 \phi_0^2 dx}{\int_0^H \rho_0 \phi_0'^2 dz} \quad (29b)$$

In the thin-pycnocline limit, $\phi_0 \sim z/H$ and $\rho_0 \sim \rho_\infty$, so

$$\int_0^H \rho_0(z) \phi_0^2 dz \sim \frac{1}{3} \rho_\infty H$$

$$\int_0^H \rho_0(z) \phi_0'^2 dz \sim \frac{1}{h} \int_0^\infty \tilde{\rho}_0 \tilde{\phi}_0'^2 d\tilde{z}$$

Equation (29b) then becomes

$$\gamma \sim \frac{\frac{1}{6} \rho_\infty h H c_0}{\int_0^\infty \tilde{\rho}_0 \tilde{\phi}'^2 d\tilde{z}} = \frac{1}{3} \beta_1 H c_0$$

and the dispersive terms in Eqs. (28) and (29a) agree. The correspondence of the nonlinear terms may be demonstrated in a similar manner. Therefore, long waves along a thin pycnocline in finite depth water behave similarly to those in shallow water.

III. Conservation Laws

It is well-known that the KdV equation has an infinite number of conserved quantities which can be used in analyzing the time evolution of the solutions. On the other hand, only four time-invariant quantities were found by Ono for the infinitely deep-water case.

Following the general procedure of Ono, we define the transformation

$$T\{A(x)\} = \int_{-\infty}^{\infty} G(x-\xi) A(\xi) d\xi \quad (30a)$$

where $A(-\infty) = A(\infty) = 0$ and

$$G(x) = \sum_{i=1,2} \frac{\beta_i}{2H_i} \coth \frac{\pi x}{2H_i} - \text{sgn}(x) \quad (30b)$$

It can be shown that

$$\int_{-\infty}^{\infty} g(x) T\{A(x)\} dx = -\int_{-\infty}^{\infty} A(x) T\{g(x)\} dx \quad (31)$$

Using Eq. (31), and integrating Eq. (25a) and its moments obtained by multiplying with powers of A, we find the following four conservation equations:

$$\frac{d}{dt} \int_{-\infty}^{\infty} A(\xi) d\xi = 0$$

$$\frac{d}{dt} \int_{-\infty}^{\infty} A^2(\xi) d\xi = 0$$

$$\frac{d}{dt} \int_{-\infty}^{\infty} \left[\frac{1}{3} A + \frac{c_0}{\alpha} A_x H\{A\}\right] dx = 0$$

$$\frac{d^2}{dt^2} \int_{-\infty}^{\infty} x A dx = 0 \quad (32)$$

The last conservation equation indicates that the velocity of the center of gravity is a time-invariant quantity.

It should be noted that Eqs. (32) are equally applicable to periodic waves if the domain of integration is taken to be one period.

IV. Solutions to the Equation

At present, analytic solutions of Eq. (25a) for arbitrary H_1 and H_2 have not been found, but it is possible to investigate the properties of the solutions numerically.

A. Normalized Equation

First, we recast the equation in normalized variables which will demonstrate more clearly the relationship of the environmental parameters and their effect on the solution. By defining the variables

$$\hat{x} = \frac{1}{\ell}(x - c_0 t), \ \hat{t} = \frac{\alpha A_m}{\ell} t, \ \hat{A} = \frac{A}{A_m} \quad (33a)$$

and the dimensionless depths

$$\hat{H}_1 = \frac{H_1}{\ell} \qquad \hat{H}_2 = \frac{H_2}{\ell} \quad (33b)$$

where

$$\ell = \frac{\beta c_0}{\alpha A_m} \qquad \beta = \beta_l + \beta_2 \tag{34}$$

and A_m is a reference value of A, we can transform Eq. (25a) to the form

$$\hat{A}_{\hat{t}} + \hat{A}\hat{A}_{\hat{x}} - \frac{\partial^2}{\partial \hat{x}^2}\int_{-\infty}^{\infty} G(\hat{x}-\xi)\hat{A}(\xi,\hat{t})\mathrm{d}\xi = 0 \tag{35a}$$

The kernel $\hat{G}$ is:

$$G(\hat{x}) = \frac{1}{2\hat{H}_l}\frac{\beta_l}{\beta}\left[\coth\frac{\pi x}{2\hat{H}_l} - \mathrm{sgn}(\hat{x})\right] + \frac{1}{2\hat{H}_l}\frac{\beta_2}{\beta}\left[\coth\frac{\pi x}{2\hat{H}_2} - \mathrm{sgn}(\hat{x})\right] \tag{35b}$$

It can be shown that the only remaining environmental parameter is β_l/β_2, so this ratio and the relative magnitude of H_l and H_2 reflect the influence of the pycnocline location.

In particular, to demonstrate the effects of finite depth, we will examine the special case where the pycnocline is located exactly halfway between the surface and the bottom. The kernel in Eq. (35a) becomes

$$G(\hat{x}) = \frac{1}{2\hat{H}_l}\left[\coth\frac{\pi\hat{x}}{2\hat{H}h_l} - \mathrm{sgn}(\hat{x})\right] \tag{36}$$

so the only arbitrary parameter remaining in the normalized equation is the depth $\hat{H}_l$.

B. Time-Dependent Waveforms

We now present the results of a series of numerical "experiments" which were performed by solving the initial value problem consisting of Eq. (35a) with various values of $\hat{H}_l$ in the kernel, Eq. (36), and both the KdV and the Benjamin/Ono (B/O) solitary waveforms as initial conditions. The calculation employed the split-step or pseudospectral method of Fornberg[5]; the periodic solutions obtained through this method will approximate the infinite domain solutions to any desired degree of accuracy if a sufficiently long period is used.

Figure 2 shows the time evolution of a waveform with a relatively short wavelength for $\hat{H}_l = 20$, where the speed of the reference frame has been chosen to be the linear long wave speed c_0. Here, it is found that the initial form evolves into a primary disturbance and a dispersive wavetrain. It is noteworthy that, near the center of the primary disturbance, the local speed defined by

$$-\frac{\partial\hat{A}}{\partial\hat{t}} \Big/ \frac{\partial\hat{A}}{\partial\hat{x}}$$

tends toward a uniform constant greater than the normalized linear speed c_0, just as would be the case for a solitary wave emerging from an initial disturbance. Similar behavior was exhibited in calculations done for other values of $\hat{H}_l$.

Figure 3 shows the time evolution of an initial disturbance with a relatively long wavelength for $\hat{H}_l = 2$ in a reference frame moving at c_0. In this case, the initial waveform decomposes into two bumps, each of which exhibits a tendency toward a uniform local speed as just described, so it appears that this calculation shows two solitary waves emerging from an initial disturbance. Furthermore, if the initial condition is symmetric about $\hat{x} = 0$, as it is here, then Eq. (35a) requires $A(-\hat{x},-\hat{t}) = A(\hat{x},\hat{t})$. and the calculation may be used backwards in time to show the two bumps merging into the initial waveform. Therefore, if the bumps are solitary waves, this calculation indicates that they may undergo interactions wherein their shapes and speeds are preserved.

The calculations for other depths produced like results, suggesting that the steady-state solutions of the finite depth equation are also solitons. These steady-state waveforms and their functional dependence on the depth are discussed in the next section.

C. Steady-State Waveforms

In the reference frame moving with a solitary wave, Eq. (35a) becomes

$$-\Delta\hat{c}\hat{A}_{\hat{x}} + \hat{A}\hat{A}_{\hat{x}} - \frac{\mathrm{d}^2}{\mathrm{d}\hat{x}^2}\int_{-\infty}^{\infty} G(\hat{x}-\xi)\hat{A}(\xi)\mathrm{d}\xi = 0 \tag{37}$$

where $\Delta\hat{c}$ is the solitary wave speed. Note that, since the $\hat{x}$ origin moves with the speed c_0 relative to the x origin, the quantity $\Delta\hat{c}$ represents the difference between the actual solitary wave speed and the linear wave speed.

Now, we can examine the functional dependence of $\hat{A}$ and $\Delta\hat{c}$ on the normalized depth $\hat{H}_l$ without any detailed knowledge of the actual solution. Equation (37) infers that

$$\hat{A} = \frac{A}{A_{\max.}} = F(\hat{x},\hat{H}_l) = F\left(\frac{\alpha A_{\max.}}{\beta c_0}x, \frac{\alpha A_{\max.}}{\beta c_0}H_l\right) \tag{38}$$

and

$$\Delta\hat{c} = \frac{\Delta c}{\alpha A_{\max.}} = \Gamma(\hat{H}_l) = \Gamma\left(\frac{\alpha A_{\max.}}{\beta c_0}H_l\right) \tag{39}$$

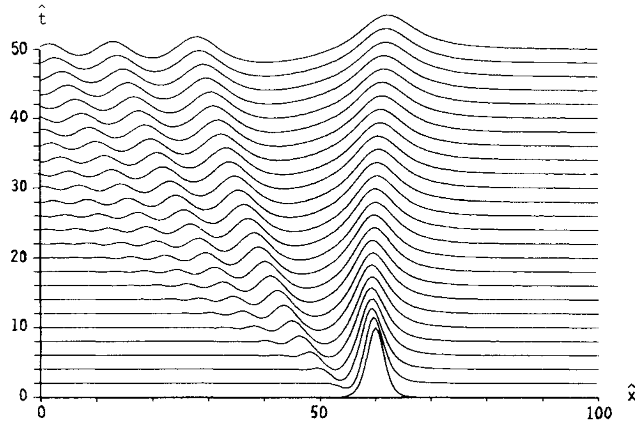

Fig. 2 Evolution of initial short wavelength disturbance in finite depth, $\hat{H}_l = \hat{H}_2 = 20$.

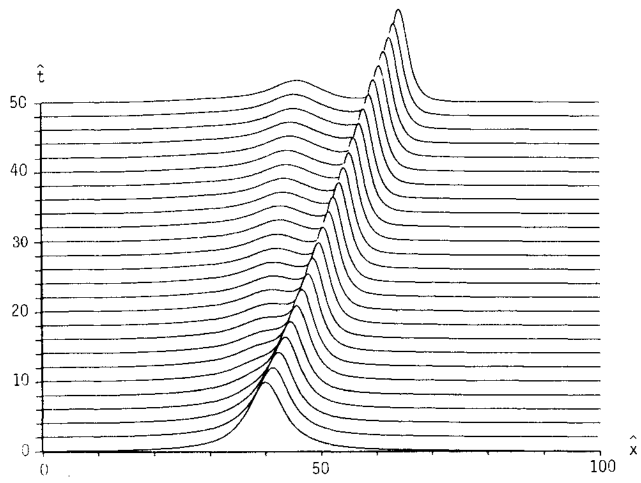

Fig. 3 Evolution of initial long wavelength disturbance in finite depth, $\hat{H}_l = \hat{H}_2 = 2$.

Also, if we define the wavelength λ by

$$\lambda = \frac{1}{A_{\max.}} \int_{-\infty}^{\infty} A(\xi)\mathrm{d}\xi = \frac{\beta c_0}{\alpha A_{\max.}} \int_{-\infty}^{\infty} F(\hat{x};\hat{H}_l)\mathrm{d}\hat{x} \qquad (40)$$

we have

$$\hat{\lambda} = \frac{\alpha A_{\max.}}{\beta c_0}\lambda = \Lambda\left(\frac{\alpha A_{\max.}}{\beta c_0} H_l\right) \qquad (41)$$

The amplitude $\hat{A}$, the wavelength $\hat{\lambda}$, and the wave speed $\Delta\hat{c}$, given by Eqs. (38, 39 and 41), demonstrate that the steady-state wave shapes given by Eq. (35a) form a two-parameter family. In particular, in the limit $H_l \to \infty$ (the infinite depth Benjamin/Ono wave), Eq. (38) becomes

$$\hat{A} = 16/(\hat{x}^2 + 16) \qquad (42)$$

with

$$\Delta\hat{c} = \frac{1}{4} \text{ and } \hat{\lambda} = 4\pi$$

Conversely, in the limit of $\hat{H}_l \to 0$ (the shallow-water KdV/Benney wave), Eq. (37) can be shown to reduce to the KdV form

$$-\Delta\hat{c}\hat{A}_{\hat{x}} + \hat{A}\hat{A}_{\hat{x}} + \frac{1}{3}\hat{H}\hat{A}_{\hat{x}\hat{x}\hat{x}} = 0 \qquad (43)$$

with the solution

$$A = \mathrm{sech}^2 \frac{\hat{x}}{2\hat{H}_l^{\,1/2}} \qquad (44)$$

where $\Delta\hat{c} = 1/3$ and $\hat{\lambda} = 4\hat{H}_l^{\,1/2}$. Thus, we expect the finite depth solitary waveforms, speeds, and wavelengths to approach those of either the KdV or B/O equations in the shallow- or deep-water limits, respectively.

We now turn our attention to the detailed solution of Eq. (37). This equation may be immediately integrated once using the boundary conditions for a solitary wave to yield the nonlinear eigenvalue problem

$$\frac{1}{2}\hat{A}^2 - \frac{\mathrm{d}}{\mathrm{d}\hat{x}}\int_{-\infty}^{\infty} [G(\hat{x}-\xi) + \Delta\hat{c}H(\hat{x}-\xi)]\hat{A}(\xi)\mathrm{d}\xi = 0 \qquad (45a)$$

with

$$\hat{A}(0) = 1 \text{ and } \hat{A}(\pm\infty) = 0 \qquad (45b)$$

For this study, the periodic version of Eqs. (45) solved with an iterative, fast Fourier transform technique, using as a convergence criterion the requirement that the sum of the nonlinear and dispersive terms in Eq. (37) must vanish everywhere; the same normalized period was used for all values of the depth, and the boundary condition at $\hat{x} = \pm\infty$ was instead applied at the half-period. As an additional check, the periodic forms of the KdV and B/O equations were also solved with the same method, and the resulting waveforms and wave speeds were found to agree with the analytic solutions for the period used. Figures 4, 5, and 6 show the results of these calculations. As expected, each of the finite-depth solitary waveforms, wave speeds, and wavelengths varies with the depth parameter in a continuous manner, bridging the two limiting situations. The second horizontal scale on the graphs of the normalized speed $\Delta\hat{c}$ and the normalized wavelength $\hat{\lambda}$ is explained in the next section.

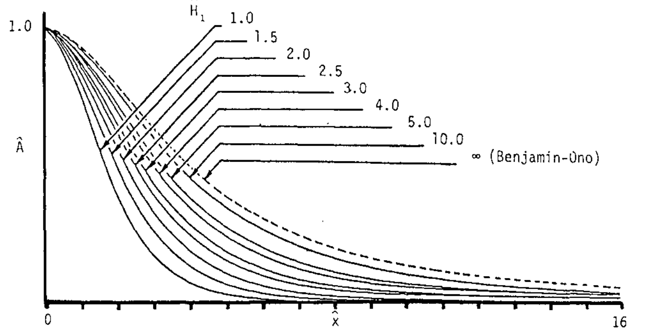

Fig. 4 Finite depth solitary waveforms for $\hat{H}_l = \hat{H}_2$.

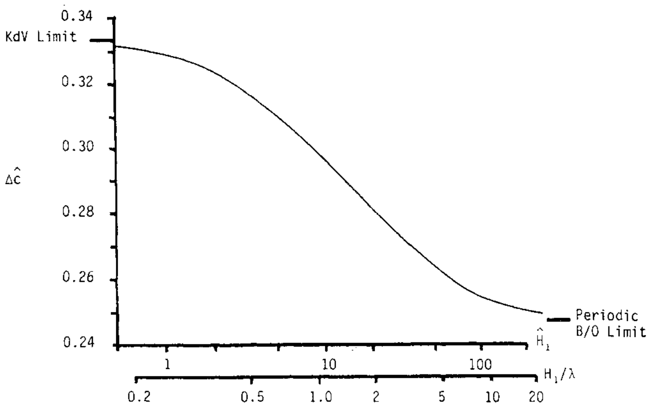

Fig. 5 Normalized wave speed vs normalized depth.

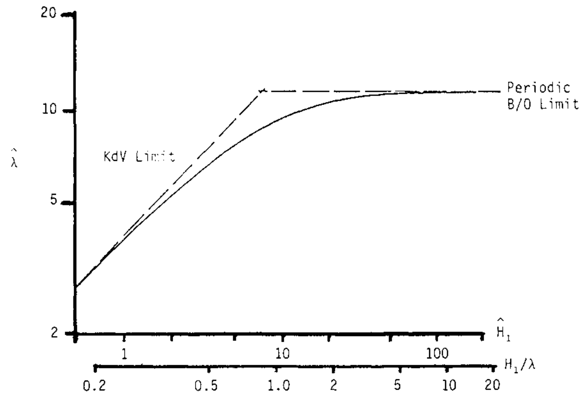

Fig. 6 Normalized wavelength vs normalized depth.

V. Discussion of Results

In order to assess properly the effects of finite depth on the steady-state waveforms, it is more desirable to return to a physical coordinate system. Thus, we introduce the following dimensionless variables:

$$\bar{\alpha} = \alpha h, \quad \bar{\beta} = \frac{\beta}{h}, \quad \bar{H}_l = \frac{H_l}{h}, \quad A_{\max.} = \epsilon c_0 h$$

Equations (39) and (41) become

$$\frac{\Delta c}{hN_{\max.}} = \epsilon\frac{\bar{\alpha}c_0}{hN_{\max.}}\Gamma\left(\epsilon\frac{\bar{\alpha}}{\bar{\beta}}H_l\right) \qquad (46)$$

$$\frac{\lambda}{h} = \frac{\Lambda\left(\epsilon\frac{\bar{\alpha}}{\bar{\beta}}H_l\right)}{\epsilon\frac{\bar{\alpha}}{\bar{\beta}}} \qquad (47)$$

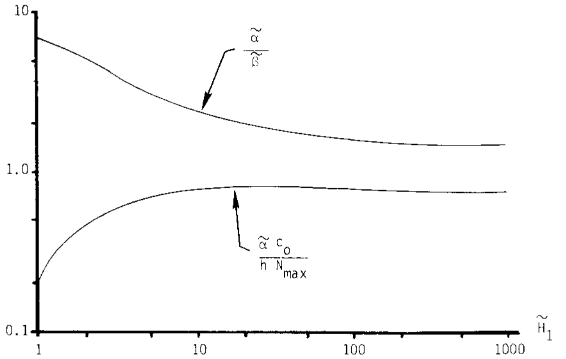

Fig. 7 Environmental parameters vs dimensionless depth.

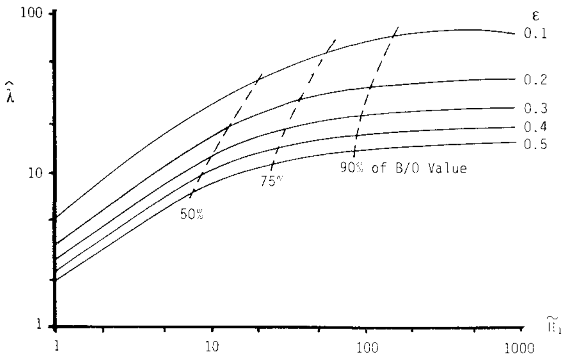

Fig. 8 Dimensionless wave speed vs dimensionless depth.

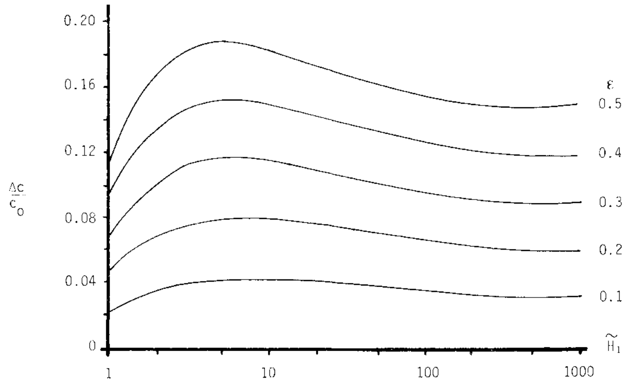

Fig. 9 Dimensionless wavelength vs dimensionless depth.

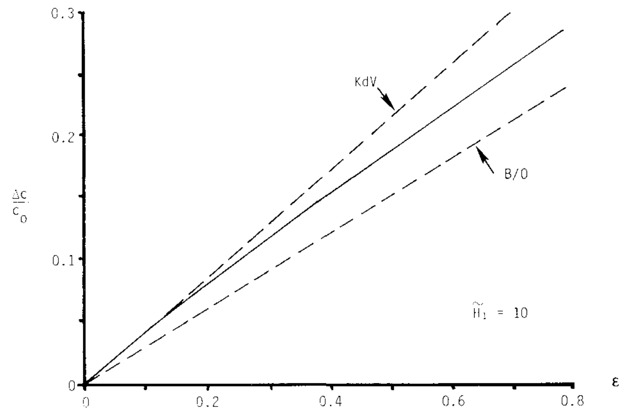

Fig. 10 Dimensionless speed vs dimensionless amplitude for $\tilde{H}_l = 10$.

where

$$N(z) = \sqrt{\frac{g\rho_0'}{2h\rho_0}}$$

is the Brunt-Vaisala frequency. It should be noted that there are just two environmental parameters in these relations: first, the quantity $\tilde{\alpha}/\tilde{\beta}$ appears primarily as a stretching factor for either the wave amplitude ϵ or the depth $\tilde{H}_l$; second, the quantity $\tilde{\alpha} c_0 / hN_{max.}$ appears as a simple multiplier for the solitary wave speed increment. As discussed in Sec. IIC, these environmental parameters depend not only on the background density distribution and the vertical mode number, but also on the depth due to the relative contribution to the various integrals from the outer region. Therefore, the overall dependence of the wave speed and wavelength on the depth will, in general, be rather complicated.

As an example, Fig. 7 shows how these environmental parameters vary with the depth for the case where the background density is given by

$$\rho_0 = \frac{\rho_l + \rho_2}{2} + \frac{\rho_l - \rho_2}{2} \tanh \tilde{z}$$

and $(\rho_l - \rho_2)/(\rho_l + \rho_2)$ is small. These values and the values of the function $\Gamma(\hat{H})$ shown in Fig. 5 may be used to find the dependence of the dimensionless solitary wave speed $\Delta c / c_0$ on the depth and wave amplitude, as illustrated in Fig. 8. It should be noted that the solitary wave speed is strongly dependent on the amplitude but only weakly dependent on the depth, because the change in c at low values of H for a given amplitude is partially counteracted by the corresponding change in $\tilde{\alpha} c_0 / hN_{max.}$.

On the other hand, the solitary wavelength is strongly dependent on the depth, as shown in Fig. 9, where the dimensionless wavelength $\tilde{\lambda} = \lambda h$ is plotted against $\tilde{H}_l$ for various amplitudes. Also shown on the figure are three dashed lines which indicate where the finite depth wavelength has fallen to various percentages of the limiting B/O wavelength.

These results clearly demonstrate that, even where the dimensionless depth $\tilde{H}_l$ is large, the wavelength of an internal solitary wave may be appreciably shorter than that of the Benjamin-Ono waveform with the same amplitude.

To illustrate the finite depth effect in a more quantitative manner, we choose $\tilde{H}_l = 10$, and plot $\Delta c / c_0$ against the wave amplitude in Fig. 10. The KdV and B/O limits are shown by the solid curve. It is rather surprising to observe that, for this given depth, the wave speed increment is actually close to the KdV limit for small amplitudes. For a larger wavelength amplitude, the results appear to approach the B/O deep water limit. But, as shown in this example, the finite depth effect is appreciable even for waves of moderate amplitude, for which the assumption of a weak nonlinearity may no longer be valid. These observations should be taken into account in future interpretations of laboratory results.

As a final remark, we note from Fig. 6 that there is a unique value of $\hat{H}_l / \hat{\lambda} = H_l / \lambda$ for each $\hat{H}_l$. Hence the normalized wave speed and wavelength may be plotted against this physical depth-to-wavelength ratio, as indicated by the second horizontal scale on Figs. 5 and 6. This helps to explain the seemingly large deviation of both the KdV and B/O theories for $\hat{H}_l \sim 10$; even though the dimensionless depth may be considered as large in this region, H_l/λ is still only 0(1).

VI. Summary

In this paper, the equation describing the propagation of a weakly nonlinear, long internal gravity wave in a stratified fluid of finite total depth has been derived. The resulting equation has been shown to reduce to the KdV equation in shallow-water limit, and to the Benjamin-Ono equation in the deep-water limit. Four conservation laws have been obtained for this equation.

The derived equation suggests the existence of a permanent waveform, which can be characterized by two parameters for the special case in which the pycnocline is located halfway between the upper and lower fluid boundaries. The two parameters have been found to be the normalized halfdepth H_l and the nondimensional amplitude ϵ. In the limiting cases of either shallow or deep water, the solution degenerates into the well-known one-parameter sech^2 or algebraic waveform, respectively. In order to solve the derived finite-depth equation a numerical code, which employs FFT techniques, was developed. Solutions generated by this code demonstrate the existence of solitary waves which exhibit solitonlike behavior. The dependence of the solitary wave speed and the wavelength as a function of the depth has also been investigated. The results indicate that, for certain laboratory and shallow ocean conditions, the finite depth effect could be appreciable. We should note that, for an arbitrary pycnocline location, parameters other than the two just mentioned would also have to be included.

References

[1]Benjamin, T.B., "Internal Waves of Finite Amplitude and Permanent Form," *Journal of Fluid Mechanics,* Vol. 25, 1966, p. 241.

[2]Benjamin, T.B., "Internal Waves of Permanent Form in Fluids of Great Depth," *Journal of Fluid Mechanics,* Vol. 29, 1967, p. 559.

[3]Benney, D.J., "Long Nonlinear Waves in Fluid Flows," *Journal of Mathematical Physics,* Vol. 45, 1966, p. 52.

[4]Davis, R.E. and Acrivos, A., "Solitary Internal Waves in Deep Water," *Journal of Fluid Mechanics,* Vol. 29, 1967, p. 593.

[5]Fornberg, B.E., "On a Fourier Method for the Integration of Hyperbolic Equations," *SIAM Journal of Numerical Analysis,* Vol. 12, 1975, p. 509.

[6]Keulegan, G.H., "Characteristics of Internal Solitary Waves," *Journal of Research of NBS,* Vol. 51, No. 3, Res. Paper 2442, 1953, p. 133.

[7]Long, R.R., "Solitary Waves in the One and Two Fluid System," *Tellus,* Vol. 8, 1965, p. 460.

[8]Long, R.R., "On the Boussinesq Approximation and Its Role in the Theory of Internal Waves," *Tellus,* Vol. 17, 1965, pp. 46-52.

[9]Ono, H., "Algebraic Solitary Waves in Stratified Fluids," *Journal of the Physical Society of Japan,* Vol. 39, No. 4, 1975, p. 1082.

[10]Whitham, G.B., *Linear and Non-Linear Waves,* John Wiley and Sons, Inc., N.Y., 1974.